On outward and inward productions in the categorical graph-grammar approach and Δ-grammars[1]

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Abstract:

We consider the relationship between three ways of defining graph derivability. That the traditional double-pushout approach and Banach's inward version are equivalent in the case of injective left-hand side is proved in a purely categorical setting. In the case of noninjective left-hand sides, equivalence can be shown in special categories if the right-hand side is injective. Both approaches have the same generative power in the category of graphs if the pushout connecting the outward production with the inward one is a pullback as well. Finally, it is shown that Banach's point of view establishes a close relationship between the categorical approach and Kaplan's Δ-grammars allowing a slight generalization of Δ-grammars and making them an operational description of the categorical approach.

1 Introduction

In the literature, you can find a lot of approaches how to generalize derivability of strings to graphs, i.e., replacing the occurrence of a left-hand side of a production by the corresponding right-hand side. In 1973, we presented an approach that is based on category theory and that uses the pushout construction to generalize the notion of concatenation [5]. In this approach, a production p is given by a pair of morphisms $p = (B^l \xleftarrow{p^l} K \xrightarrow{p^r} B^r)$ with common domain; derivability is defined by two pushouts concatenating a host graph with the left-hand side and the right-hand side of the production, respectively. The main advantage of the categorical approach to graph grammars is that it does not refer to special properties of graphs and, therefore, the results can easily be extended to other categories that have pushouts, such as high-level replacement systems [4], graphs with a structure on the set of labels [8], and hierarchical graphs the nodes and edges of which are labeled with graphs again [9].

Our double-pushout approach follows formal language theory: A derivation step starts with looking for a redex $g^l : B^l \to G^l$ and then B^l is replaced by B^r. In term graph rewriting, however, the steps occur in the reverse order: Gluing some new structure into the graph is done first, then edges are redirected, and finally garbage is removed. In a recently presented lecture, R. Banach could show that this order is also possible in the double-pushout approach [1, 2]. In his productions, the arrows go inwards: $q = (B^l \xrightarrow{q^l} Q \xleftarrow{q^r} B^r)$, where B^l and B^r

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are the same graphs as in our approach and the morphisms q^l and q^r are connected to p^l and p^r by a pushout construction. Banach could show that under certain assumptions each inward derivation step unambiguously corresponds to an outward derivation step. The direction of the arrows leads us to use the terms "inward" and "outward" referring to Banach's approach and to the original form, respectively.

The first aim of this paper is to generalize Banach's result by proving it in a purely categorical setting, thus making it applicable to cases more general than usual graphs: Replacing an outward derivation step by an inward one is possible in any category that has pushouts. The opposite direction does not hold in general, but it can be shown under certain assumptions. In addition to Banach's result, the equivalence result still holds as long as one side of the production is injective. (Banach required the left-hand side to be injective.) If, however, the pushout connecting the outward and the inward production is a pullback as well, the equivalence of both approaches can be established at least for some interesting categories, e.g., the category of graphs. Finally, we establish a correspondence between Banach's inward productions and Kaplan's Δ-productions making the latter an operational interpretation of the double-pushout approach. This result allows the idea behind Δ-productions, i.e., representing a production by a tripartite graph, to be applied to hierarchically labeled graphs and other graph-like structures. Since a tripartite graph can be implemented by marking nodes and edges in a suitable way, it seems that this correspondence may simplify implementation of the categorical approach.

The rest of this paper is organized as follows: In the next section, we recall some basic notions of the categorical graph-grammar approach. We also summarize the properties of colimits as far as we need them. In Section 3, we prove the relationship between the inward approach and the outward one. Section 4 shows how to apply the results concerning injective productions to Δ-grammars.

2 Basic notions and results

As we have already mentioned, derivability can be defined by two pushouts concatenating the host graph with the left-hand side and the right-hand side of the production, resp.:

Definition 2.1 Let $p = (B^l \xleftarrow{p^l} K \xrightarrow{p^r} B^r)$ be a production. We call G^r derivable from G^l via p: $G^l \stackrel{p}{\Longrightarrow} G^r$ (with redex $g^l : B^l \to G^l$) if and only if there exists a morphism $g : K \to C$ such that in the following diagram both squares are pushouts:

$$\begin{tikzcd}
B^l \arrow[d, "g^l"'] \arrow[dr, phantom, "PO"] & K \arrow[l, "p^l"'] \arrow[r, "p^r"] \arrow[d, "g"] \arrow[dr, phantom, "PO"] & B^r \arrow[d, "g^r"] \\
G^l & C \arrow[l, "\bar{p}^l"] \arrow[r, "\bar{p}^r"'] & G^r
\end{tikzcd}$$

An advantage of the double-pushout approach is its symmetry. Applying it from left to right means generating graphs; in syntax analysis, however, we use productions from right to left. The operations we have to perform are the same.

Definition 2.2 With each outward production $p = (B^l \xleftarrow{p^l} K \xrightarrow{p^r} B^r)$, we associate an inward production $q = (B^l \xrightarrow{q^l} Q \xleftarrow{q^r} B^r)$, where Q is the pushout object in

$$\begin{tikzcd} K \arrow[r, "p^r"] \arrow[d, "p^l"'] \arrow[dr, phantom, "PO"] & B^r \arrow[d, "q^r"] \\ B^l \arrow[r, "q^l"'] & Q \end{tikzcd}$$

This definition is asymmetrical: If an outward production is given, the associated inward production can always be constructed unambiguously, but not vice versa. There may be several outward productions the given inward production is associated with, or it may happen that the inward production is not associated with any outward production at all. Our definition excludes the latter case restricting discussion to inward productions constructed from an outward one.

Definition 2.3 Let $q = (B^l \xrightarrow{q^l} Q \xleftarrow{q^r} B^r)$ be an inward production. We call G^r derivable from G^l via q: $G^l \stackrel{q}{\Longrightarrow} G^r$ (with redex $g^l : B^l \to G^l$) if and only if there exists a morphism $g^r : B^r \to G^r$ such that in the following diagram both squares are pushouts

$$\begin{tikzcd} B^l \arrow[r, "q^l"] \arrow[d, "g^l"'] \arrow[dr, phantom, "PO"] & Q \arrow[d, "\bar{g}"] \arrow[dr, phantom, "PO"] & B^r \arrow[l, "q^r"'] \arrow[d, "g^r"] \\ G^l \arrow[r] & D & G^r \arrow[l] \end{tikzcd}$$

We often use (p^l, p^r) and (q^l, q^r) as abbreviations for the outward and inward production, respectively.

A problem of the double-pushout approach is the necessity to construct the "pushout complement", e.g., after determining a redex g^l in the outward version, we have to look for a $g : K \to C$ such that G^l is the pushout object of g and p^l. (There is an asymmetry between the outward and inward approach. Using the outward one, we have to construct the pushout complement in the middle, no matter in which direction we apply the production. Contrary to this, the inward approach causes us to look for a pushout complement when constructing the right-most or left-most morphism, respectively.) The following lemma establishes a relationship between pushouts and coproducts; it is useful in constructing pushout complements if g' and p are given [3, 5]:

Lemma 2.4 In any category that has coproducts and pushouts, we have:

$$\begin{tikzcd} \cdot \arrow[r, "p"] \arrow[d, "g"'] & \cdot \arrow[d, "g'"] & \cdot \arrow[l, "\bar{p}"'] \arrow[dl, "g' \cdot \bar{p}"] \\ \cdot \arrow[r, "p'"'] & \cdot \end{tikzcd}$$

(a)

$$\begin{tikzcd} \cdot \arrow[r, "p"] \arrow[d, "g"'] & \cdot \arrow[d, "g'"] \\ \cdot \arrow[r, "p'"] & \cdot \\ \cdot \arrow[u, "\bar{g}"] \arrow[ur, "\bar{g}'"'] & \end{tikzcd}$$

(b)

(a) If $g' \cdot p = p' \cdot g$ and if there exists a $\bar{p}$ such that $(p, \bar{p})$ is a coproduct, then $g' \cdot p = p' \cdot g$ is a pushout diagram if and only if $(p', g' \cdot \bar{p})$ is a coproduct.

(b) If p and g' are given such that there exists a $\bar{g}'$ with $(\bar{g}', g')$ being a coproduct, then the coproduct $(\bar{g}, g)$ yields a pushout diagram $g' \cdot p = p' \cdot g$.

In $\mathcal{S}et$, (b) ensures existence of the pushout complement if g' is injective, whereas (a) allows construction for injective p if the following identification condition is fulfilled additionally:

Criterion 2.5 In $\mathcal{S}et$, $g' : B \to G$ satisfies the identification condition with respect to $p : K \to B$ if

$$(\forall x, y \in B)(g'(x) = g'(y) \Rightarrow x = y \vee x, y \in p[K])$$

If, however, both p and g' are not injective, we can split them into surjections and injections. In the case of two surjections, at least one pushout complement exists, and all solutions can be characterized by

$$\begin{aligned} (\forall x, y)(g'(x) = g'(y) \quad \Rightarrow \quad & x = y \quad \vee \\ & (\exists k, k')(p(k) = x \wedge p(k') = y \wedge g(k) = g(k'))) \end{aligned}$$

The double-pushout lemma then allows composing the separately constructed pushouts [6, p. 147]:

Lemma 2.6 In any category with pushouts, a commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, "="] & B \arrow[r, "f'"] \arrow[d, "q"'] \arrow[dr, phantom, "="] & B' \arrow[d, "q'"] \\ C \arrow[r, "p"] & D \arrow[r, "p'"] & D' \end{tikzcd}$$

satisfies the following properties:

(a) If $q \cdot f = p \cdot g$ and $q' \cdot f' = p' \cdot q$ are pushout diagrams, then $q' \cdot (f' \cdot f) = (p' \cdot p) \cdot g$ is a pushout diagram, too.

(b) If $q \cdot f = p \cdot g$ and $q' \cdot (f' \cdot f) = (p' \cdot p) \cdot g$ are pushout diagrams, then $q' \cdot f' = p' \cdot q$ is a pushout diagram.

In general, the first part need not be a pushout if the whole diagram and the second part are pushouts, but in the category $\mathcal{S}et$, this does hold if additionally f' is injective.

Up to now, our definitions do not refer to special properties of graphs and, therefore, can be applied to all categories that have pushouts. In this paper, however, we are mainly interested in graphs:

Definition 2.7 A graph G is a quadruple $G = (G_E, G_V, s_G, t_G)$ with G_E and G_V being sets and $s_G, t_G : G_E \to G_V$ being mappings.

G_E and G_V denote the set of edges and nodes (vertices), respectively. $s_G(e)$ denotes the source node of an edge e and $t_G(e)$ its target node. Considering more than one graph, we distinguish their constituents by indices referring to the denotations of the graphs, e.g., B^l_V is the set of nodes of graph B^l, etc.

Definition 2.8 Let G and H be graphs. A graph morphism $f : (G_E, G_V, s_G, t_G) \to (H_E, H_V, s_H, t_H)$ is a pair $f = (f_E : G_E \to H_E, f_V : G_V \to H_V)$ of set morphisms such that

$$f_V \cdot s_G = s_H \cdot f_E \quad \wedge \quad f_V \cdot t_G = t_H \cdot f_E$$

Limits and colimits in the category $\mathcal{G}raph$ can separately be constructed for nodes and edges in the category $\mathcal{S}et$. In this paper, we need coproducts, pushouts and pullbacks. The pushout complement exists if Crit. 2.5 is satisfied for both nodes and edges, but we have to prevent the edges from losing their source or target nodes:

Criterion 2.9 In $\mathcal{G}raph$, $g' : B \to G$ satisfies the dangling condition with respect to $p : K \to B$ if

$$s_G[G_E \setminus g'_E[B_E]] \subseteq (G_V \setminus g'_V[B_V]) \cup g'_V p_V[K_V]$$

$$t_G[G_E \setminus g'_E[B_E]] \subseteq (G_V \setminus g'_V[B_V]) \cup g'_V p_V[K_V]$$

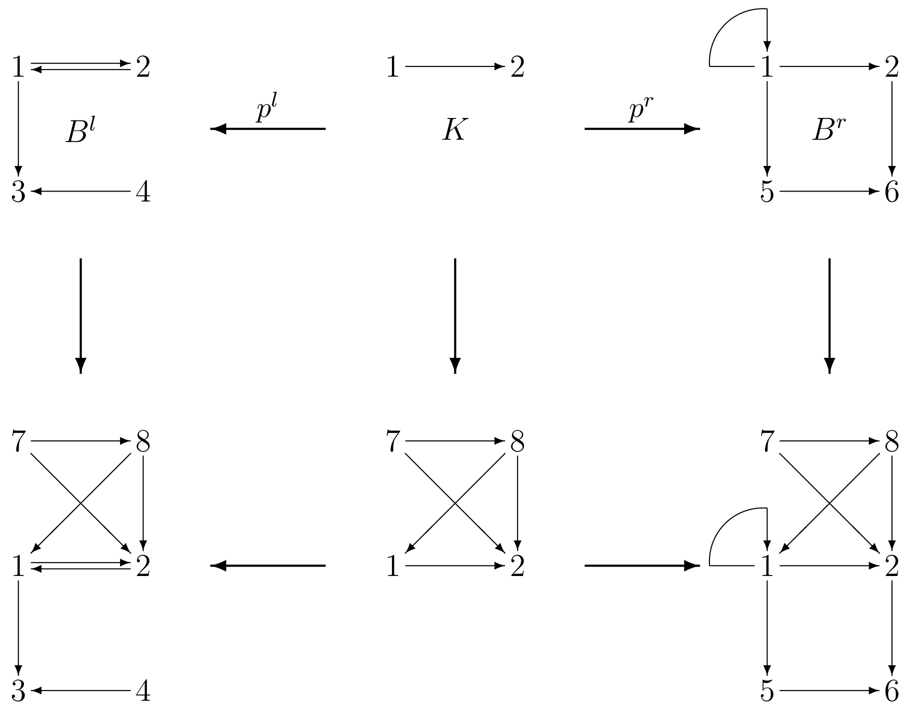

Fig. 1: Injective example using an outward production

If this criterion is satisfied, separate construction of C_V and C_E yields a graph C that is the pushout complement of $K \to B$ and $B \to G$. We have already shown in our seminal paper on graph grammars [5] that a unique C exists if p is injective and the identification condition as well as the dangling condition hold, together referred to as the gluing condition. Although that proof was restricted to injective p, the condition is applicable to the noninjective case, too; the resulting C, however, is ambiguous.

Let us consider an example. Fig. 1 describes a derivation step using an injective outward production. The production is depicted in the upper part of the figure. The nodes are numbered in such a way that the numbers also denote the morphisms: a node is mapped to the node with the same number, e.g., $p^l(1) = 1$. Then, mapping of edges is implicitly defined. Since all morphisms are injective and the gluing condition is satisfied, the derived graph exists and is unambiguous. The same graph can be derived using the associated inward production (Fig. 2).

3 Outward vs. inward productions

Banach's main result is that in the category $\mathcal{G}raph$, we can replace the outward double-pushout construction by an inward one and vice versa, if the gluing condition holds and p^l is injective. The first part of this result, however, can be obtained without any restrictions

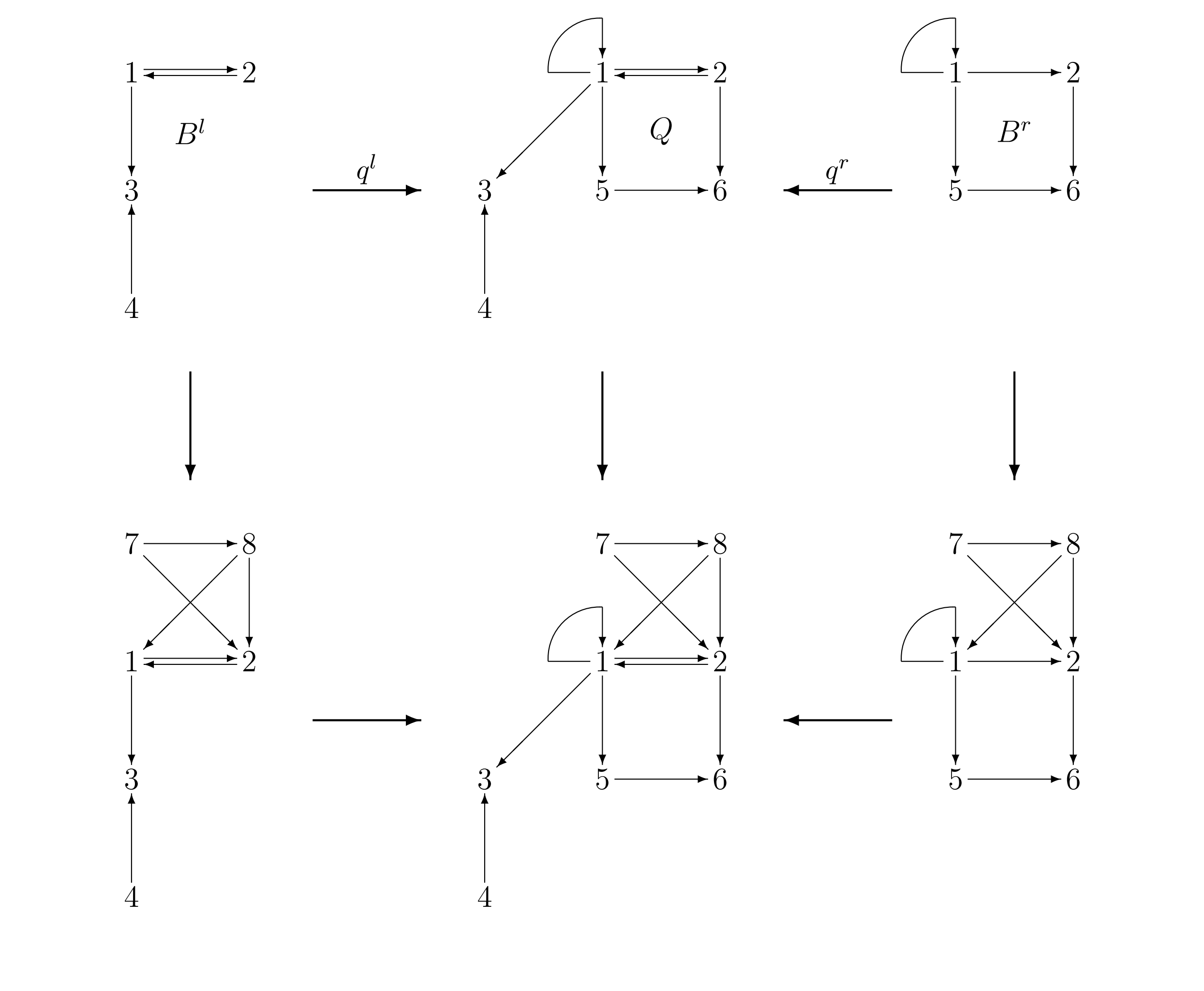

Fig. 2: Injective example using an inward production

in every category that has pushouts[3]:

Theorem 3.1 If we have a derivation step $G^l \overset{p}{\Longrightarrow} G^r$ via an outward production p in a category that has pushouts, then $G^l \overset{q}{\Longrightarrow} G^r$ does also hold, where q is the inward production associated with p.

Proof: We consider an outward production (p^l, p^r) and the inward production (q^l, q^r) associated with it, i.e., $q^l \cdot p^l = q^r \cdot p^r$ is a pushout diagram (Fig. 3). We assume that G^r is derivable from G^l by applying the outward production at redex g^l, i.e., the squares $g^l \cdot p^l = \bar{p}^l \cdot g$ and $g^r \cdot p^r = \bar{p}^r \cdot g$ are pushout diagrams, too. Then, we define D by constructing the pushout square $\bar{q}^l \cdot \bar{p}^l = \bar{q}^r \cdot \bar{p}^r$ from $\bar{p}^l$ and $\bar{p}^r$. By the pushout property of Q,

$$(\bar{q}^l \cdot g^l) \cdot p^l = \bar{q}^l \cdot \bar{p}^l \cdot g = \bar{q}^r \cdot \bar{p}^r \cdot g = (\bar{q}^r \cdot g^r) \cdot p^r$$

[3]This is no restriction since the double-pushout approach makes no sense in a category that does not have pushouts.

$$
\begin{tikzcd}
 & K \arrow[rr, "p^r"] \arrow[dl, "p^l"'] \arrow[dd, "g"] & & B^r \arrow[dl, "q^r"] \arrow[dd, "g^r"] \\
B^l \arrow[rr, "q^l"'] \arrow[dd, "g^l"'] & & Q \arrow[dd, "\bar{g}"] & \\
 & C \arrow[rr, "\bar{p}^r"] \arrow[dl, "\bar{p}^l"'] & & G^r \arrow[dl, "\bar{q}^r"] \\
G^l \arrow[rr, "\bar{q}^l"'] & & D &
\end{tikzcd}
$$

Fig. 3: Pushouts connecting inward and outward derivations

yields a unique $\bar{g}$ with

$$\bar{q}^l \cdot g^l = \bar{g} \cdot q^l \quad \wedge \quad \bar{q}^r \cdot g^r = \bar{g} \cdot q^r.$$

Now, we use Lemma 2.6: With $g^r \cdot p^r = \bar{p}^r \cdot g$ and $\bar{q}^r \cdot \bar{p}^r = \bar{q}^l \cdot \bar{p}^l$ being pushouts, $(\bar{q}^r \cdot g^r) \cdot p^r = \bar{q}^l \cdot (\bar{p}^l \cdot g)$ also is a pushout. Furthermore, $(\bar{g} \cdot q^r) \cdot p^r = \bar{q}^l \cdot (g^l \cdot p^l)$ is the same pushout (up to isomorphism) because of $\bar{g} \cdot q^r = \bar{q}^r \cdot g^r \wedge g^l \cdot p^l = \bar{p}^l \cdot g$. Thus, the second half of this pushout, namely $\bar{g} \cdot q^l = \bar{q}^l \cdot g^l$, is a pushout, too. Analogously, we see that $\bar{q}^r \cdot g^r = \bar{g} \cdot q^r$ is a pushout. Putting these pushouts together, we get derivability via q. $\square$

In *Graph*, this means that if the gluing condition holds in the outward derivation step, then the associated inward production can also be applied, i.e., an analogous gluing condition holds in Banach's approach. If p^l is not injective, the outward production, nevertheless, may be applied if the pushout complement exists, but C and G^r need not be unambiguous. Theorem 3.1 says that at least these graphs G^r can also be derived by the inward production. Conversely, it is easy to prove that every inward derivation step corresponds to an outward one in an HLR1-category [4], if p^l and p^r are in the distinguished class M of morphisms that is typical of an HLR1-category. A more precise discussion, however, shows that a weaker condition is sufficient:

Definition 3.2 Let $\mathcal{C}$ be a class of categories with pushouts such that in each category, there is a distinguished class M of morphisms satisfying the following properties:

(a) If $f : A \to B$ is in M, $g : A \to C$ is any morphism and $q \cdot f = p \cdot g$ is the pushout of (f, g), then p is in M, too.

(b) If p is in M and $p' \cdot p$ has a pushout complement, i.e., there exists a pushout diagram $r \cdot s = p' \cdot p$, then r and s are unambiguous (up to isomorphism).

(c) If in the following diagram both squares are pushout diagrams, p is in M and $p' \cdot p$ has a pushout complement, then $f' \cdot f$ has a pushout complement, too.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, "="] & B \arrow[r, "f'"] \arrow[d, "q"] \arrow[dr, phantom, "="] & B' \arrow[d, "q'"] \\ C \arrow[r, "p"] & D \arrow[r, "p'"] & D' \end{tikzcd}$$

Lemma 3.3 Both $\mathcal{S}et$ and $\mathcal{G}raph$ together with injections as the distinguished class of morphisms satisfy the requirements of Def. 3.2.

Proof: In $\mathcal{S}et$, property (a) is well-known. A pushout complement exists if and only if Criterion 2.5 is satisfied. Therefore, property (b) immediately follows from Lemma 2.4(a) since injections have unambiguous coproduct complements. To prove the last property, we have to consider two different elements in B with $f'(x) = f'(y)$ and to show $x, y \in f[A]$. Of course, we have $p'q(x) = p'q(y)$. If $q(x) = q(y)$ holds, $x, y \in f[A]$ follows from the pushout property of the left-hand part of the diagram. Otherwise, we get $q(x), q(y) \in p[C]$ by assumption and $q(x), q(y) \in p[C] \cap q[B]$ yields $x, y \in f[A]$.

In the case of graphs, we additionally need Criterion 2.9 to be inherited backward from $p' \cdot p$ to $f' \cdot f$. We consider an edge $e \in B'_E \setminus f'_E[B_E]$. From the pushout property of the right-hand part of the diagram, we get $q'(e) \in D'_E \setminus p'_E[D_E]$. Since q' is a graph morphism, the assumption yields

$$s_{D'} q'_E(e) = q'_V s_{B'}(e) \in (D'_V \setminus p'_V[D_V]) \cup p'_V p_V[C_V]$$

The first case together with the pushout property of the right-hand part of the diagram yield $s_{B'}(e) \in B'_V \setminus f'[B_V]$, whereas the second part and the pushout property of the whole diagram result in $s_{B'}(e) \in f'_V f_V[A_V]$. □

This lemma says that the following theorem is applicable to graph productions:

Theorem 3.4 If in a category of class $\mathcal{C}$, p is an outward production with $p^l \in M$, q is the associated inward production and if we have a redex $g^l : B^l \to G^l$, then $G^l \overset{p}{\Longrightarrow} G^r$ holds if and only if $G^l \overset{q}{\Longrightarrow} G^r$.

Proof: The direction from left to right is a consequence of Theorem 3.1. Conversely, i.e., we assume $G^l \overset{q}{\Longrightarrow} G^r$, three of the pushouts in the cube of Fig. 3 exist:

$$q^r \cdot p^r = q^l \cdot p^l \qquad \bar{g} \cdot q^l = \bar{q}^l \cdot g^l \qquad \bar{g} \cdot q^r = \bar{q}^r \cdot g^r$$

This means that q^r is in M since p^l is and that $\bar{g} \cdot q^r$ has a pushout complement that is unambiguous. Therefore, the inward derivation step, existence of which we have assumed, yields an unambiguous G^r if the production and the redex are given. From condition (c) in Def. 3.2, we have that $g^l \cdot p^l$ also has a unique pushout complement. Thus we have a unique $\bar{G}^r$ derivable by the outward derivation step. But Theorem 3.1 says that this $\bar{G}^r$ can be derived by an inward step, too. This results in $\bar{G}^r = G^r$. $\square$

This result essentially coincides with Banach's [1]. The advantage of our proof in a purely categorical setting is that we can apply it to more general structures, e.g., high-level replacement systems [4] and hierarchical graphs [9]. Making use of the symmetry in Fig. 3, we can analogously prove a result allowing ambiguous derivation steps[4]:

Theorem 3.5 If in a category of class $\mathcal{C}$, p is an outward production with $p^r \in M$, q is the associated inward production and if we have a redex $g^l : B^l \to G^l$, then $G^l \stackrel{p}{\Longrightarrow} G^r$ holds if and only if $G^l \stackrel{q}{\Longrightarrow} G^r$.

If both p^l and p^r are not in M, however, then the inward approach may derive more G^r than the outward one is able to derive from the same G^l applying the production at the same redex. Since pushouts in $\mathcal{G}raph$ can be constructed for nodes and edges separately, it is sufficient to give an example in the category $\mathcal{S}et$. We start with the following outward production (p^l, p^r):

$$(p1) \qquad B^l : \{1, [2,3], 4\} \xleftarrow{\quad p^l \quad} K : \{1,2,3,4\} \xrightarrow{\quad p^r \quad} B^r : \{[1,2], 3, 4\}$$

As before, the elements of the sets are denoted by numbers, which define the mapping as well. Square brackets indicate an element that is the image of more than one element, e.g., $p^l(2) = p^l(3) = [2,3]$.

We embed B^l into a G^l by lumping together the elements denoted by 1 and 4 (Fig. 4). As we have already mentioned, we can choose between different pushout complements. In our case, both g_1 and g_2 yield G^l at the left-hand side. Therefore, we obtain two different results at the right-hand side.

The inward production (q^l, q^r) associated with (p^l, p^r) can be found by constructing the pushout of (p^l, p^r):

$$(q1) \qquad B^l : \{1, [2,3], 4\} \xrightarrow{\quad q^l \quad} Q : \{[1,2,3], 4\} \xleftarrow{\quad q^r \quad} B^r : \{[1,2], 3, 4\}$$

It is easy to see that our previous results, G_1^r and G_2^r, can also be derived using this inward production (Fig. 5). In this case, however, we get a third alternative G_3^r, which is not derivable using the outward approach!

[4]A derivation step is called ambiguous if applying a production to a graph at a fixed redex yields different results.

$$\begin{tikzcd}
B^l : \{1, [2,3], 4\} \arrow[d, "g^l"'] & K : \{1,2,3,4\} \arrow[l, "p^l"'] \arrow[r, "p^r"] \arrow[d, "g_1"] & B^r : \{[1,2], 3, 4\} \arrow[d, "g_1^r"] \\
G^l : \{[1,4], [2,3]\} & C_1 : \{[1,4], 2, 3\} \arrow[l] \arrow[r] & G_1^r : \{[1,2,4], 3\}
\end{tikzcd}$$

$$\begin{tikzcd}
B^l : \{1, [2,3], 4\} \arrow[d, "g^l"'] & K : \{1,2,3,4\} \arrow[l, "p^l"'] \arrow[r, "p^r"] \arrow[d, "g_2"] & B^r : \{[1,2], 3, 4\} \arrow[d, "g_2^r"] \\
G^l : \{[1,4], [2,3]\} & C_2 : \{[1,4], [2,3]\} \arrow[l] \arrow[r] & G_2^r : \{[1,2,3,4]\}
\end{tikzcd}$$

Fig. 4: Noninjective example using outward productions

A detailed analysis of the example shows that the gluing object K in production $(p1)$ is not fine-grained enough to treat element 4 differently from both 2 and 3. If, however, we look for the most fine-grained outward production the inward production $(q1)$ is associated with, we have to take the pullback of (q^l, q^r)[5]:

$$\begin{tikzcd}
K^b \arrow[r, "p^{br}"] \arrow[d, "p^{bl}"'] \arrow[dr, phantom, "PB"] & B^r \arrow[d, "q^r"] \\
B^l \arrow[r, "q^l"'] & Q
\end{tikzcd}$$

In our example, the pullback construction yields

$$(p2) \qquad B^l : \{1, [2,3], 4\} \xleftarrow{p^{bl}} K^b : \{a, b, c, d, e\} \xrightarrow{p^{br}} B^r : \{[1,2], 3, 4\}$$

with
$$\begin{aligned}
p^{bl}(a) = p^{bl}(b) &= 1 & p^{br}(a) = p^{br}(c) &= [1,2] \\
p^{bl}(c) = p^{bl}(d) &= [2,3] & p^{br}(b) = p^{br}(d) &= 3 \\
p^{bl}(e) &= 4 & p^{br}(e) &= 4
\end{aligned}$$

[5]If we construct (q^l, q^r) as the pushout of (p^l, p^r) and (p^{bl}, p^{br}) as the pullback of (q^l, q^r), then (q^l, q^r) is associated with (p^{bl}, p^{br}), too, because of the Galois correspondence between pullbacks and pushouts [6, p. 147]

$$\begin{tikzcd}
B^l : \{1, [2,3], 4\} \arrow[r, "q^l"] \arrow[d, "g^l"'] & Q : \{[1,2,3], 4\} \arrow[d, "\bar{g}"] & B^r : \{[1,2], 3, 4\} \arrow[l, "q^r"'] \arrow[d, "q^r_1"] \\
G^l : \{[1,4], [2,3]\} \arrow[r] & D : \{[1,2,3,4]\} & G^r_1 : \{[1,2,4], 3\} \arrow[l]
\end{tikzcd}$$

$$\begin{tikzcd}
B^l : \{1, [2,3], 4\} \arrow[r, "q^l"] \arrow[d, "g^l"'] & Q : \{[1,2,3], 4\} \arrow[d, "\bar{g}"] & B^r : \{[1,2], 3, 4\} \arrow[l, "q^r"'] \arrow[d, "q^r_2"] \\
G^l : \{[1,4], [2,3]\} \arrow[r] & D : \{[1,2,3,4]\} & G^r_2 : \{[1,2,3,4]\} \arrow[l]
\end{tikzcd}$$

$$\begin{tikzcd}
B^l : \{1, [2,3], 4\} \arrow[r, "q^l"] \arrow[d, "g^l"'] & Q : \{[1,2,3], 4\} \arrow[d, "\bar{g}"] & B^r : \{[1,2], 3, 4\} \arrow[l, "q^r"'] \arrow[d, "q^r_3"] \\
G^l : \{[1,4], [2,3]\} \arrow[r] & D : \{[1,2,3,4]\} & G^r_3 : \{[1,2], [3,4]\} \arrow[l]
\end{tikzcd}$$

Fig. 5: Noninjective example using inward productions

Checking that all derivation steps of Fig. 5 can be realized by this production is left to the reader. (Constructing the pushout complement leads to six different solutions, four of which finally result in the same G^r.)

Theorem 3.6 If we have an outward production (p^l, p^r) and its associated inward production (q^l, q^r) such that $q^l \cdot p^l = q^r \cdot p^r$ is a bicartesian square, i.e., both a pushout and a pullback, then

$$G^l \overset{p}{\Longrightarrow} G^r \quad \Longleftrightarrow \quad G^l \overset{q}{\Longrightarrow} G^r$$

holds in $\mathcal{S}et$ and $\mathcal{G}raph$.

Proof: We have to prove only the direction from right to left, since the other direction follows from Theorem 3.1. Given an inward derivation step, we construct the object C of

Fig. 3 as the pullback object of $(\bar{q}^l, \bar{q}^r)$. Therefore, C is the subset of the Cartesian product $G^l \times G^r$ on which $\bar{q}^l \cdot \bar{p}^l$ and $\bar{q}^r \cdot \bar{p}^r$ coincide, with $\bar{p}^l$ and $\bar{p}^r$ being the projections; this means that C consists of all element pairs (g_l, g_r) with $\bar{q}^l(g_l) = \bar{q}^r(g_r)$. Similarly, K is (isomorphic to) the subset of $B^l \times B^r$ on which $q^l \cdot p^l = q^r \cdot p^r$ holds. From the pullback property of C, we get a unique $g : K \to C$ such that $g^l \cdot p^l = \bar{p}^l \cdot g$ and $g^r \cdot p^r = \bar{p}^r \cdot g$ commute[6]. We have to prove that these diagrams are pushouts. Since the diagram is symmetric, we can restrict discussion to the left-hand side. We have to show that G^l is isomorphic to the quotient $B^l \uplus C/_{\equiv}$ with $\equiv$ being the reflexive, transitive closure of

$$x \sim y \quad :\Leftrightarrow \quad (\exists k \in K)(x = p^l(k) \wedge y = g(k)).$$

In the present case, the generating relation $\sim$ is given by[7]

$$b_l = p^l(b_l, b_r) \sim g(b_l, b_r) = (g^l(b_l), g^r(b_r)) \qquad (b_l \in B^l, b_r \in B^r)$$

where the last equality follows from

$$g(b_l, b_r) = (\bar{p}^l g(b_l, b_r), \bar{p}^r g(b_l, b_r)) = (g^l p^l(b_l, b_r), g^r p^r(b_l, b_r)) = (g^l(b_l), g^r(b_r)).$$

We first show that two different elements $(g_l, g_r), (g_l', g_r')$ of C are thrown together by $\bar{p}^l$ if and only if there exist two elements $(b_l, b_r), (b_l', b_r') \in K$ with

$$g(b_l, b_r) = (g_l, g_r) \wedge g(b_l', b_r') = (g_l', g_r') \wedge p^l(b_l, b_r) \equiv p^l(b_l', b_r').$$

Since C and K are pullbacks, we have

$$\begin{array}{lcl} \bar{p}^l(g_l, g_r) = \bar{p}^l(g_l', g_r') & \Leftrightarrow & g_l = g_l' \\ p^l(b_l, b_r) = p^l(b_l', b_r') & \Leftrightarrow & b_l = b_l'. \end{array}$$

This reduces the condition to be shown to:

$$(g_l, g_r), (g_l, g_r') \in C \wedge g_r \neq g_r' \Rightarrow (g_l, g_r) \equiv (g_l, g_r').$$

By construction of C, we get from the left-hand side of this condition

$$\bar{q}^r(g_r) = \bar{q}^r \bar{p}^r(g_l, g_r) = \bar{q}^l \bar{p}^l(g_l, g_r) = \bar{q}^l(g_l) = \bar{q}^l \bar{p}^l(g_l, g_r') = \bar{q}^r \bar{p}^r(g_l, g_r') = \bar{q}^r(g_r').$$

The following diagram taking into consideration only the elements of interest illustrates the situation:

[6] Since the morphisms no longer are injective, the construction we have used in the proof of Theorem 3.4 may yield different pushout complements that do not fit in with one another to form a double-pushout.

[7] For convenience, we do not write $g((b_l, b_r))$.

$$
\begin{tikzcd}
 & \cdot \arrow[rr, "p^r"] \arrow[dl, "p^l"'] \arrow[dd, "g"] & & b_r, b'_r \arrow[dl, "q^r"'] \arrow[dd, "g^r"] \\
\cdot \arrow[rr, "q^l"] \arrow[dd, "g^l"'] & & \cdot \arrow[dd, "\bar{g}"] & \\
 & (g_l, g_r), (g'_l, g'_r) \arrow[rr, "\bar{p}^r"] \arrow[dl, "\bar{p}^l"'] & & g_r, g'_r \arrow[dl, "\bar{q}^r"] \\
g_l = g'_l \arrow[rr, "\bar{q}^l"'] & & \bar{q}^r(g_r) = \bar{q}^r(g'_r) &
\end{tikzcd}
$$

Since $\bar{q}^r \cdot g^r = \bar{g} \cdot q^r$ is a pushout, we can choose two different elements $b_r, b'_r \in B^r$ with

$$g^r(b_r) = g_r \wedge g^r(b'_r) = g'_r \wedge q^r(b_r) = q^r(b'_r).$$

The pushout property of $q^l \cdot p^l = q^r \cdot p^r$ yields existence of two elements $(b_l, b_r), (b'_l, b'_r)$ in K with $p^l(b_l, b_r) = p^l(b'_l, b'_r)$. This is equivalent to $b_l = b'_l$. We now have

$$(g^l(b_l), g_r) = g(b_l, b_r) \sim p^l(b_l, b_r) = b_l = p^l(b_l, b'_r) \sim g(b_l, b'_r) = (g^l(b_l), g'_r)$$

and we can choose b_l such that we get $g^l(b_l) = g_l$ and therefore $(g_l, g_r) \sim (g'_l, g'_r)$. This choice is possible because of the following argument: If we had chosen another b_l, an element b''_l must exist with $g^l(b''_l) = g_l \neq g^l(b_l)$ and $q^l(b''_l) = q^l(b_l)$ because of the pushout property of $\bar{g} \cdot q^l = \bar{q}^l \cdot g^l$; therefore, $(b''_l, b_r), (b''_l, b'_r)$ are in K, too, and we can choose b''_l instead of b_l.

Next, we show that $b_l, b'_l \in B^l \wedge b_l \neq b'_l \wedge g^l(b_l) = g^l(b'_l)$ results in $b_l \equiv b'_l$, i.e., we have to show existence of a b_r such that

$$b_l \sim (g^l(b_l), g^r(b_r)) = (g^l(b'_l), g^r(b_r)) \sim b'_l.$$

If $q^l(b_l) = q^l(b'_l)$, this follows from the pushout property of $q^l \cdot p^l = q^r \cdot p^r$ similarly to the arguments of the first case. Otherwise, we use the pushout property of $\bar{g} \cdot q^r = \bar{q}^r \cdot g^r$.

If $g_l \in G^l \setminus g^l[B^l]$, then $\bar{q}^l(g_l)$ cannot be in $\bar{g}[Q]$ and must be in $\bar{q}^r[G^r]$ because of the given pushouts. Thus, we have a $g_r \in G^r$ with $\bar{q}^r(g_r) = \bar{q}^l(g_l)$, and (g_l, g_r) is in C by definition. Therefore, $g_l = \bar{p}^l(g_l, g_r) \in \bar{p}^l[C]$. Conversely, let be $g_l \in G^l \setminus \bar{p}^l[C]$. In this case, definition of C yields $\bar{q}^l(g_l) \notin \bar{q}^r[G^r]$, and from the pushouts of the inward derivation steps, we have $\bar{q}^l(g_l) \in \bar{g}[Q]$ and $g_l \in g^l[B^l]$.

This completes the proof in the case of $\mathcal{S}et$. In $\mathcal{G}raph$, the result holds both for nodes and edges. Furthermore, (C_E, C_V) can unambiguously be made a graph by constructing s_C and t_C using the pullback property of C_V. $\square$

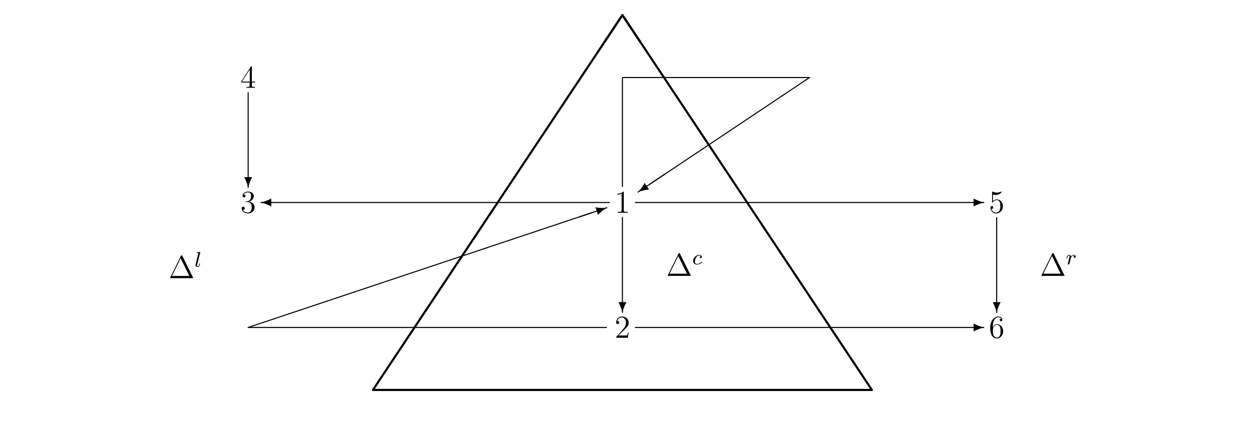

Fig. 6: Δ-production

4 Application to Δ-grammars

The operational approaches to graph rewriting describe the process of replacing a subgraph by another one in an immediately implementable way. The Δ-grammars, which were presented by S.M. Kaplan et al. [7], incorporate all relevant items of a production into one tripartite graph:

Definition 4.1 A Δ-production is a tripartite graph

$$\Delta = (\Delta_E^l \cup \Delta_E^c \cup \Delta_E^r, \Delta_V^l \cup \Delta_V^c \cup \Delta_V^r, s, t)$$

where the following conditions hold:

$$\begin{array}{ll} (a) & s, t : \Delta_E^l \cup \Delta_E^c \cup \Delta_E^r \rightarrow \Delta_V^l \cup \Delta_V^c \cup \Delta_V^r \\ (b) & e \in \Delta_E^l \Rightarrow s(e) \notin \Delta_V^r \wedge t(e) \notin \Delta_V^r \\ (c) & e \in \Delta_E^r \Rightarrow s(e) \notin \Delta_V^l \wedge t(e) \notin \Delta_V^l \\ (d) & e \in \Delta_E^c \Rightarrow s(e) \in \Delta_V^c \wedge t(e) \in \Delta_V^c \end{array}$$

Usually, Δ^c is depicted within a triangle, and Δ^l and Δ^r are drawn on the left and on the right, respectively. (Δ^x is shorthand for the edges Δ_E^x and the nodes Δ_V^x. Please note that in general Δ^l and Δ^r are not graphs, but Def. 4.1 ensures that $\Delta^l \cup \Delta^c$ and $\Delta^c \cup \Delta^r$ are graphs if we suitably restrict s, t.) Δ^l is the fragment of the graph removed during applying the production, Δ^r is the new graph replacing Δ^l. Δ^c denotes a subgraph that is identified, but not changed during rewriting. It is common to both the left-hand side and the right-hand side of the production.[8] Fig. 6 depicts the Δ-version of the production we have already seen in Fig. 1 and in Fig. 2. Δ^c consists of the nodes within the triangle and the edges not leaving it, whereas an edge a part of which is drawn outside the triangle belongs to Δ^l or to Δ^r, respectively. E.g., the edges from 2 to 1 and from 1 to 3 are part of Δ^l.

[8] Kaplan's productions have two further components, a negative application condition and a textually expressed guard; both are omitted here.

$$\begin{tikzcd}
\Delta^l \arrow[r] \arrow[dr] & \Delta^l \amalg \Delta^c \arrow[d, "g^l"'] & \Delta^c \arrow[l] \arrow[r] \arrow[d, "g"] & \Delta^c \amalg \Delta^r \arrow[d, "g^r"] \\
 & G = \Delta^l \amalg \Delta^c \amalg R & \Delta^c \amalg R \arrow[l] \arrow[r] & \Delta^c \amalg \Delta^r \amalg R = H \\
(a) & & R \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}$$

$$\begin{tikzcd}
\Delta^c \arrow[r] \arrow[d] & \Delta^c \amalg \Delta^r \arrow[d] & \Delta^r \arrow[l] \arrow[dl] \\
(b) \quad \Delta^l \amalg \Delta^c \arrow[r] & \Delta^l \amalg \Delta^c \amalg \Delta^r &
\end{tikzcd}$$

Fig. 7: Δ-production and the double-pushout approach

Definition 4.2 If a Δ-production is given and we have an injective graph morphism

$$g^l : (\Delta^l_E \cup\!\!\!\cdot\, \Delta^c_E, \Delta^l_V \cup\!\!\!\cdot\, \Delta^c_V, s_p, t_p) \to G$$

where s_p, t_p are the restrictions of s, t to $\Delta^l_E \cup\!\!\!\cdot\, \Delta^c_E$, then H is called derivable from G with $\Delta : G \stackrel{\Delta}{\Longrightarrow} H$ if H is constructed as follows:

$$\begin{array}{lll}
H_E & := & (G_E \setminus g^l[\Delta^l_E]) \cup\!\!\!\cdot\, \Delta^r_E \\
H_V & := & (G_V \setminus g^l[\Delta^l_V]) \cup\!\!\!\cdot\, \Delta^r_V \\
s_H(e) & := & \begin{cases} s_G(e) & \text{if } e \in G_E \setminus g^l[\Delta^l_E] \\ s(e) & \text{if } e \in \Delta^r_E \wedge s(e) \in \Delta^r_V \\ g^l(s(e)) & \text{if } e \in \Delta^r_E \wedge s(e) \in \Delta^c_V \end{cases} \\
t_H(e) & & \text{analogously}
\end{array}$$

Construction of s_H and t_H ensures that H is a graph again.

Theorem 4.3 In the case of injective redices Δ-derivability and double-pushout derivability are equivalent.

Proof: If a production $\Delta = (\Delta^l, \Delta^c, \Delta^r)$ is given, we can construct an injective outward production $p = (\Delta^l \amalg \Delta^c \xleftarrow{p^l} \Delta^c \xrightarrow{p^r} \Delta^c \amalg \Delta^r)$ with p^l and p^r being the natural coproduct morphisms. (We use coproduct notation instead of disjoint union to keep discussion more general.) For injective redices, Fig. 7(a) shows that $G \stackrel{p}{\Longrightarrow} H \Leftrightarrow G \stackrel{\Delta}{\Longrightarrow} H$ is an immediate consequence of Lemma 2.4(b).[9] Furthermore, Fig. 7(b) and Lemma 2.4(a) yield that the Q of Def. 2.3 is isomorphic to $\Delta^l \amalg \Delta^c \amalg \Delta^r$ since the coproduct operator is associative up to

[9] R is the remaining part of the host graph.

isomorphism. Conversely, you can easily see that the Q of an injective inward production can be partitioned such that the resulting Δ-production is equivalent to the double-pushout production. $\square$

This means that we can implement the double-pushout approach (at least in the case of injective productions) storing only one graph, namely Q, together with its partition.

A closer look at Fig. 7(a) allows for generalizing Δ-derivability such that noninjective redices are taken into consideration. Since Lemma 2.4(a) does not assume g' itself to have a coproduct complement, but only $g' \cdot \bar{p}$ and since $g^l[\Delta^l \amalg \Delta^c] = g^l[\Delta^l] \amalg g^l[\Delta^c]$ (Criterion 2.5), we immediately get that the formulas of Def. 4.2 still hold if g^l is injective only on Δ^l.

5 Conclusion

We have considered the inward and outward derivability in the categorical graph-grammar approach and we could prove Banach's equivalence result in a purely categorical framework. This technique has led us to a more general result, establishing the equivalence in the case of noninjective left-hand (or right-hand) sides, too. In the case that both sides of a production are not injective, we could show equivalence if the outward production and the inward one are connected by a pushout diagram that additionally is a pullback. Nevertheless, it is a disappointing proof since it does not make use of categorical methods. On the other hand, it is easy to verify that the result does also hold in some other categories of interest, e.g., in the category $\mathcal{S}etincl$ of sets with set inclusion as morphisms. The question of how to prove the result in a categorical framework is open.

An advantage of discussing Δ-derivability in the categorical framework is that we can apply it to hierarchically labeled graphs, i.e., graphs the nodes and edges of which are labeled with graphs again. (In [9], we have shown how to construct pushouts in such a category.) In this case, we have to partition the labels in the center, too. Of course, we have some problems to draw larger graphs in the Δ-version; coloring can solve the problem. In Fig. 8, (a) depicts an example taken from [9, p. 269], and (b) is the pushout object of (p^l, p^r). The colors are indicated by upper indices: l and r denote nodes and edges that are part of Δ^l or of Δ^r, respectively, and c denotes elements that are part of Δ^c. $(\bullet, \perp, \circ)$ is short-hand for a graph consisting of three isolated nodes labeled with $\bullet^l$, $\perp^c$, and $\circ^r$, respectively. In a forthcoming paper, we plan to discuss this concept in the context of actor grammars, where we have to label the nodes with term graphs. Remembering that Banach started with observing a difference between the double-pushout approach and term graph rewriting, this may come full circle.

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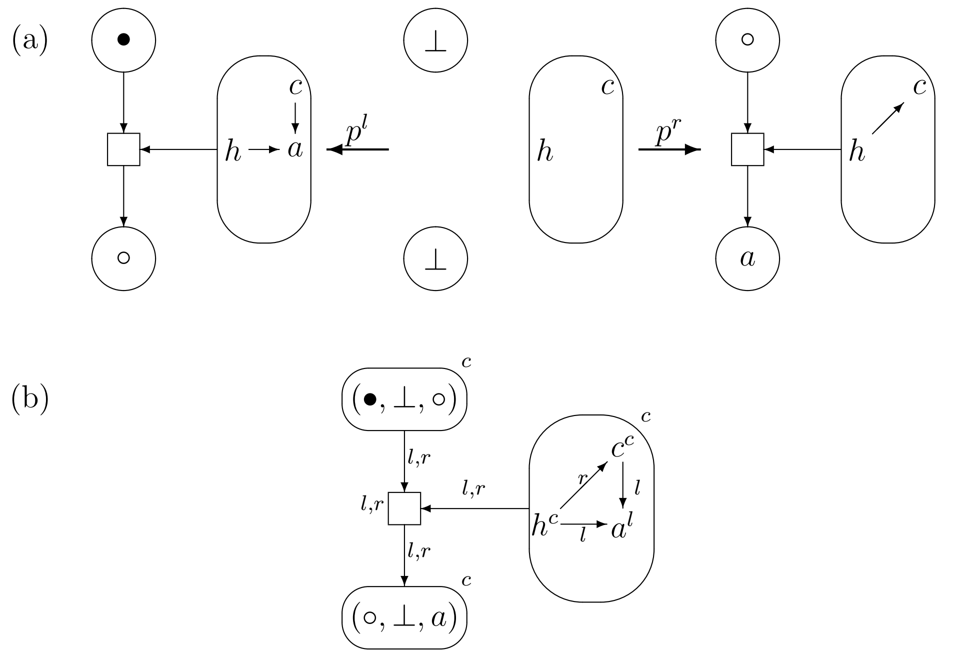

Fig. 8: Hierarchical graph production and its tripartite version

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